Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 14 (2003), 111–114.

PROBLEM SECTION

Edited by Nenad P. Cakić and Milan J. Merkle

We publish research problems in all areas of Mathematics that fall in one of the following categories:

a) Research problems or conjectures with a solution not known to the proposer. These problems are marked with an asterisk after their number.

b) Research problems seeking a new, more elegant solution

c) Inquiries about references and state of the art regarding a particular problem.

Problems should be submitted in a form that is easy to understand to a nonspecialist in a field. If using special terms or notations can not be avoided, they should be defined in a statement of a problem. A problem may be accompanied by a short comment (addressed primarily to specialists) that explains why the solution could be of an interest.

Solutions should be worked out in all reasonable details.

Correspondence regarding Problem Section should be sent to:
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• **Problem 50** proposed by Slavko Simić and Vladimir Janković, Belgrade, Serbia and Montenegro.

If real numbers a, b, c satisfy: a) $a+b+c=2$, b) $bc+ca+ab \geq abc+1$, then the inequality

$$(ax+by+cz)^2 \geq 4\big((1-a)\,yz+(1-b)\,zx+(1-c)\,xy\big)$$

holds for each real x, y, z.

• **Problem 51** proposed by Péter Ivády, Budapest, Hungary.

Show that for $x>0$,

$$\sqrt{1-\exp\left(-\frac{x^2}{\sqrt{x^2+1}}\right)} < \tanh x < \sqrt[3]{1-\exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)}$$

holds.

• **Problem 52** proposed by Mihály Bencze, Brasov, Romania.

Let be $f: \mathbb{R} \to \mathbb{R}$ a differentiable function such that

$$f'(x) + a\,f(x) = e^{-bx^2}$$

for all $x \in \mathbb{R}$ and $a, b > 0$. Compute $\lim\limits_{x\to\infty} f(x)$.

• **Problem 53** proposed by Mihály Bencze, Brasov, Romania.

The function $f: \mathbb{R} \to \mathbb{R}$ is integrable and $0 < \alpha \le 1$. Compute

$$\lim_{n\to+\infty} \sum_{k=1}^{n} k^{-\alpha} f\left(\frac{k}{n}\right)\ \mathrm{tg}\left(\frac{k^\alpha}{n}\right).$$

• **Problem 54** proposed by Mihály Bencze, Brasov, Romania.

Prove the inequality:

$$\int_0^{\pi/2} \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x > \frac{11\,\pi}{162} + \frac{38}{81}\,.$$

• **Problem 55** proposed by Slavko Simić, Mathematical Institute, Belgrade, Serbia and Montenegro.

Prove that the following precise inequality holds for $x \ge 1/4$:

$$\log\left(\frac{12\,x^2 + 6x + 1 - \dfrac{1}{60\,x^2+2}}{12\,x^2 - 6x + 1 - \dfrac{1}{60\,x^2+2}}\right) < \frac{1}{x} < \log\left(\frac{12\,x^2 + 6x + 1 - \dfrac{1}{60\,x^2+1}}{12\,x^2 - 6x + 1 - \dfrac{1}{60\,x^2+1}}\right).$$

• **Problem 56*** proposed by Slavko Simić, Mathematical Institute, Belgrade, Serbia and Montenegro.

Prove or disprove that the inequality $b > 1 + \log^\beta b$, holds for each $b > 1$ if and only if $\beta \in [1, 5/2]$.

• **Problem 57*** proposed by Slavko Simić, Mathematical Institute, Belgrade, Serbia and Montenegro.

Let $f(x) \in C_I^\infty$. It is said that f have *smoothness* of order k if the first k derivatives of f are strictly positive on $I := [0, \infty)$. Find the order of smoothness of $f(x)$ such that $1/f(x)$ is convex for all sufficiently large x.

• **Problem 58** proposed by Ó. CIAURRI RAMÍREZ, Universidad de La Rioja, Logroño, Spain and J. L. DÍAZ-BARRERO, Univesitat Politècnica de Catalunya, Barcelona, Spain.

Let $\mathbb{A} = \{a_n\}_{n\geq 0}$ be a sequence of real numbers. We define

$$T^1(\mathbb{A}) = \{T^1_n(\mathbb{A})\}_{n\geq 0} = \left\{\sum_{k=0}^{n} F_{n-k+1}\, a_k\right\},$$
$$R^1(\mathbb{A}) = \{R^1_n(\mathbb{A})\}_{n\geq 0} = \left\{\sum_{k=0}^{n} (-2)^{n-k}(k+1)\, a_{k+1}\right\},$$

and , for $m > 1, T^m(\mathbb{A}) = T^1\big(T^{m-1}(\mathbb{A})\big)$ and $R^m(\mathbb{A}) = R^1\big(R^{m-1}(\mathbb{A})\big)$. Show that

$$T^m(\mathbb{F}) = \frac{1}{m!}\, R^m(\mathbb{F}),$$

where $\mathbb{F} = \{F_{n+1}\}_{n\geq 0}$ and F_n is the n^{th} FIBONACCI number defined by $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

SOLUTIONS

• **Problem 44** proposed by ZDRAVKO F. STARC, Vršac, Serbia and Montenegro.

Let F_n be FIBONACCI numbers. Prove that

$$F_1F_2F_3 + F_2F_3F_4 + \cdots + F_{n-1}F_nF_{n+1} = \frac{1}{3}\, F_{n+1}^{\,3} - \frac{1}{30}\, F_{3n-1} + (-1)^{n+1}\,\frac{1}{5}\, F_{n-2} - \frac{1}{2}\,.$$

Solution by proposer. Letting $x = F_1, F_2, \ldots, F_{n-1}$ and $y = F_2, F_3, \ldots, F_n$, respectively, in the identity

$$xy\,(x+y) = \frac{1}{3}\,\big((x+y)^3 - x^3 - y^3\big),$$

adding and by virtue of

$$F_1^{\,3} + F_2^{\,3} + \cdots + F_n^{\,3} = \frac{1}{10}\,\big(F_{3n+2} + (-1)^{n+1}\, 6\, F_{n-1} + 5\big)$$

we obtain the required identity.

• **Problem 46*** proposed by ALEXANDRY LUPAS, Faculty of sciences, University of Sibiu, Romania.

Let a be a fixed positive number. Find all sequences $(a_n)_{n=1}^{\infty}$ such that $1 \leq a_n \leq an$ and

$$\lim_{n\to\infty} \frac{1}{\Gamma(1+\alpha_n)} \int_0^{nx} e^{-t} t^{\alpha_n}\, \mathrm{d}t = 0, \qquad \forall x \in [0, a].$$

Solution by SLAVKO SIMIĆ, Mathematical Institute, Belgrade, Serbia and Montenegro.

There is no such sequence. For the proof we need the following lemma.

Lemma 1. *If* $\alpha_n \to \infty$, *then*

$$\int_0^1 (te^{-t})^\alpha \mathrm{d}t \sim \frac{1}{2}\sqrt{\frac{2\pi}{\alpha_n}}\, e^{-\alpha_n}.$$

This is a classical theory of LAPLACE's integral, since the point $t = 1$ of maximum of the integrand coincides with the end of the interval of integration (see [**1**] M. V. FEDORIOK: *Metod Perevala.* Nauka, Moskva 1977, p. 42).

Note that the sequence $(\alpha)_{n\geq 1}$ has to be unbounded and also that the condition cited problem should be valid for $x = a$.

But, by Lemma 1 and STIRLING's formula, we get

$$\frac{4}{\Gamma(1+\alpha_n)} \int_0^{nx} e^{-t} t^{\alpha_n}\,\mathrm{d}t = [t \to \alpha_n t] = \frac{\alpha_n^{\alpha_n+1}}{\Gamma(1+\alpha_n)} \int_0^{na/\alpha_n} (te^{-t})^{\alpha_n} \mathrm{d}t$$

$$\geq \frac{\alpha_n^{\alpha_n+1}}{\Gamma(1+\alpha_n)} \int_0^1 (te^{-t})^{\alpha_n} \mathrm{d}t \sim \frac{\alpha_n^{\alpha_n+1}}{\Gamma(1+\alpha_n)} \cdot \frac{1}{2}\sqrt{\frac{2\pi}{\alpha_n}}\, e^{-\alpha_n} \to \frac{1}{2} \quad (\alpha_n \to \infty).$$